Journal of Engineering Mathematics, Vol. 10, No. 3, July 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

The stability of inviscid plane Couette flow in the presence of random fluctuations

M. J. MANTON

Department of Mathematics, Monash University, Clayton, Victoria, Australia.

L. A. MYSAK*

Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada.

(Received June 23, 1975)

SUMMARY

The stability of a plane parallel flow which varies randomly across a channel is investigated. The mean velocity profile corresponds to plane Couette flow which is stable in the absence of fluctuations. The mean component of an infinitesimal disturbance in the flow is governed by an equation which is analogous to the Rayleigh equation arising in the classical stability analysis of an inviscid flow. The flow is found to be unstable when the correlation scale of the fluctuations in the basic flow is small compared with the channel width.

1. Introduction

Plane Couette flow of an inviscid fluid is theoretically stable with respect to infinitesimal disturbances [1]. Similarly, analysis of a linear profile in a viscous fluid shows it to be stable [2]. On the other hand, such a flow is found experimentally to become unstable at a Reynolds number of a few thousand. But any physical realisation of a plane Couette flow is not straightforward and is liable to introduce random fluctuations into the basic flow. For example, fluctuations would be produced by any eccentricity of the walls of a large radius annulus. In the present work, we therefore consider the stability of a plane parallel basic flow which varies randomly across the channel but has a linear mean velocity. The fluid is taken to be inviscid, and so any unstable disturbance found by this analysis has a corresponding unstable disturbance in a viscous fluid at high Reynolds number. On the other hand, a viscous fluid might support further unstable disturbances at a finite Reynolds number [3].

The random system is analysed by the methods developed to study random wave fields [4, 5]. A second order ordinary differential equation, analogous to the Rayleigh equation, is derived to describe the behaviour of the mean vorticity of an infinitesimal disturbance. It is found that the flow is unstable provided that the correlation scale of the fluctuations in the basic flow is small compared with the channel width. This is not surprising when it is recalled that a deterministic basic profile is unstable if it contains an inflexion point [3].

* Also, Institute of Oceanography.

Thus the smaller the correlation scale of the fluctuations, the higher the probability of an inflexion point in the profile of any given realisation of the flow, and so the higher the probability of instability in the mean.

2. Equation for mean disturbance vorticity

We consider the two-dimensional motion of an incompressible inviscid fluid in a channel of width H. Cartesian co-ordinates (x, y), normalized by H, are taken such that x varies along the channel and the channel walls are at $y = 0, 1$. There is a steady basic flow which is a random function of y such that the mean velocity varies linearly; that is, the basic velocity along the channel is $U_0U(y)$ where

$$U(y) = y + \varepsilon V(y). \tag{2.1}$$

Here V is a statistically homogeneous zero-mean random function with a variance of unity, and so

$$EV = 0 \text{ and } EV^2 = 1 \tag{2.2}$$

where E is the averaging operator which corresponds physically to taking an ensemble average. Hence $\varepsilon^2 U_0^2$ is the mean square fluctuation in the basic velocity profile. A measure of the correlation scale of the fluctuations is given by H/σ where

$$\sigma^2 = E(dV/dy)^2 = -R''(0), \tag{2.3}$$

where $R(x) = EV(x + y)V(y)$ is the correlation function of V.

The stability of the basic flow is studied by the introduction of an infinitesimal disturbance into the flow. The motion of the two-dimensional disturbance is governed by the linearised vorticity equation [2]

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\Psi_0 - U''\frac{\partial \Psi_0}{\partial x} = 0 \tag{2.4}$$

where t is time normalised by H/U_0, $'$ denotes differentiation with respect to y, and Ψ_0 is a streamfunction normalised by U_0H. The function Ψ_0 is random because U in (2.4) is random. The boundary condition that there is no flow through the walls of the channel implies that

$$\Psi_0 = 0 \text{ on } y = 0, 1. \tag{2.5}$$

We now consider the solution of (2.4)–(2.5) subject to the deterministic initial condition $\Psi_0 = \Psi_0(x, y, 0)$ at $t = 0$; that is, no randomness is introduced by the initial state of the disturbance. The Fourier transform of Ψ_0 is defined as

$$\Psi(y; k, \omega) = \int_{-\infty}^{\infty} dx \int_0^{\infty} dt\, \Psi_0(x, y, t) \exp\{-i(kx + \omega t)\}, \tag{2.6}$$

where Ψ is analytic in the lower half ω-plane because $\Psi_0 \equiv 0$ for $t < 0$. Hence (2.4)–(2.5) may be transformed to obtain

$$(U - c)(\Psi'' - k^2\Psi) - U''\Psi = f(y; k) \tag{2.7}$$

with $\Psi = 0$ on $y = 0, 1$, where $\omega = -kc$ and

$$f = (-i/k)\left(\frac{d^2}{dy^2} - k^2\right)\int_{-\infty}^{\infty} dx\, e^{-ikx}\Psi_0(x, y, 0).$$

The homogeneous part of (2.7) is the Rayleigh system which is found by taking a normal mode decomposition of (2.4)–(2.5).

By defining the operators

$$L = (y - c)\left(\frac{d^2}{dy^2} - k^2\right), \quad M = V\left(\frac{d^2}{dy^2} - k^2\right) - V'' \tag{2.8}$$

and using (2.1), equation (2.7) may be written as

$$L\Psi + \varepsilon M\Psi = f.$$

This may be decomposed into its mean and random components, namely

$$L\psi + \varepsilon EM\phi = f \tag{2.9a}$$

and

$$L\phi + \varepsilon M\psi + \varepsilon(M\phi - EM\phi) = 0, \tag{2.9b}$$

where $\Psi = \psi + \phi$, $E\Psi = \psi$ and $E\phi = 0$.

Formally solving (2.9b) for ϕ and substituting into (2.9a), we find

$$L\psi + \varepsilon EM \sum_{n=0}^{\infty} [-\varepsilon(I - E)L^{-1}M]^{n+1}\psi = f. \tag{2.10}$$

Frisch [4] suggests that the series in (2.10) converges provided that ε is sufficiently small. Then the behaviour of the mean disturbance is described well by retaining only the first term of the series; this is the "first order smoothing approximation"

$$L\psi - \varepsilon^2 EML^{-1}M\psi = f, \text{ with } \psi = 0 \text{ on } y = 0, 1. \tag{2.11}$$

The stability of the basic flow is determined by the asymptotic behaviour of ψ as $t \to \infty$. But the only growing components of the solution come from the spectrum of the associated homogeneous problem [3]. Hence it is necessary only to consider the eigenvalue problem

$$L\psi - \varepsilon^2 EML^{-1}M\psi = 0, \text{ with } \psi = 0 \text{ on } y = 0, 1; \tag{2.12}$$

where it is seen from (2.6) and (2.7) that ψ is analytic in the upper half c-plane when k is positive. Using (2.2), (2.3) and (2.8), we find that the explicit form of (2.12) is

$$\begin{aligned}\{(y - c)^2 - \varepsilon^2\}(\psi'' - k^2\psi) &= \varepsilon^2\sigma^2\psi(y) + \varepsilon^2(y - c)\int_0^1 dx\, G(x, y)\\ &\times \{R''(y - x)[\psi'' - k^2\psi(x)] - R^{\mathrm{IV}}(y - x)\psi(x)\}/(x - c),\\ &\text{with } \psi = 0 \text{ on } y = 0, 1;\end{aligned} \tag{2.13}$$

where $R(y)$ is the correlation function as defined following (2.3), and

$$G(x, y) = \begin{cases} \sinh kx \sinh k(1 - y)/k \sinh k, & \text{for } 0 < x < y < 1 \\ \sinh ky \sinh k(1 - x)/k \sinh k, & \text{for } 0 < y < x < 1 \end{cases}$$

is proportional to the Green's function of L.

Although the integrand of the last term in (2.13) is singular at $x = c$, the integral itself is multiplied by $y - c$ and so the term is zero at $y = c$. The equation has singular points at $y = c \pm \varepsilon$. Away from these points the terms of order ε^2 may be neglected and the solution is unaffected by the random fluctuations. The additional terms must be considered when $y - c$ is of order ε; but then the integral term is of order $\varepsilon^3 \log \varepsilon$ whereas the others are of order ε^2. It appears therefore that, to order ε^2, the integral term of (2.13) does not contribute significantly to the behaviour of ψ. Hence we take the equation for the mean component of the disturbance to be

$$\{(y - c)^2 - \varepsilon^2\}(\psi'' - k^2\psi) - \varepsilon^2\sigma^2\psi(y) = 0, \text{ with } \psi = 0 \text{ on } y = 0, 1. \tag{2.14}$$

It is shown below in § 3 that (2.14) can also be derived from physical considerations under the assumption that the correlation scale of the fluctuations is small compared with the channel width.

3. Physical interpretation of vorticity equation

The homogeneous part of (2.7), the Rayleigh equation, may be considered as a linearised equation for the conservation of vorticity in a reference frame moving at the phase speed of the disturbance. The first term represents the longitudinal advection of vorticity by the basic flow, while the second term accounts for the lateral advection of vorticity by the disturbance. A corresponding interpretation can be made of the equation when decomposed into its mean and random components. Thus the homogeneous part of (2.9a), which describes the mean vorticity balance, may be written explicitly as

$$(y - c)(\psi'' - k^2\psi) + \varepsilon EV(\phi'' - k^2\phi) - \varepsilon EV''\phi = 0. \tag{3.1}$$

Here the first two terms account for the longitudinal advection of vorticity by the total basic flow. Only the random part of the basic flow has a vorticity gradient, and hence the mean lateral transport of vorticity is supported by the random disturbance velocity advecting the random basic vorticity. The first order smoothing approximation, resulting in (2.11), corresponds to the neglect of second order random terms in (2.9b). The approximate equation for the fluctuating vorticity balance is therefore

$$(y - c)(\phi'' - k^2\phi) - \varepsilon V(\psi'' - k^2\psi) - \varepsilon V''\psi = 0. \tag{3.2}$$

Equation (3.2) gives an explicit expression for the fluctuating vorticity of the disturbance $(\phi'' - k^2\phi)$ in terms of the mean disturbance (ψ) and the basic velocity $(y + \varepsilon V)$. Thus the second term in (3.1) can be written in terms of the local values of ψ, V and their derivatives. On the other hand, the fluctuating disturbance streamfunction (ϕ which is proportional to the lateral velocity) is found by inverting (3.2) and so it depends upon the integral over the channel of the fluctuating basic velocity and its curvature. If the correlation scale of the basic fluctuations is small compared with the channel width then such integrals approximate weighted averages of the zero-mean random functions. (These "averages" are weighted by the deterministic parts of the integrands.) We therefore assume that the effect in (3.1) of the smoothed function ϕ is small compared with that of the locally determined function $\phi'' - k^2\phi$. Hence the last term in (3.1) is neglected, leaving the mean vorticity equation implying that there is no net longitudinal transport of vorticity. Now the substitution of (3.2) into the reduced (3.1) gives the equation of motion (2.14).

Comparison of (2.14) with the Rayleigh equation (2.7) shows that the effects of the fluctuations in the basic flow are to change the net longitudinal advection velocity $(y - c)$ by the factor $\{1 - \varepsilon^2/(y - c)^2\}$ and to introduce an apparent mean curvature of $-\varepsilon^2 EVV''/(y - c)$. These minus signs occur in these correction terms because, as shown by (3.2), any change in the fluctuating vorticity is compensated by a corresponding change in the mean vorticity.

4. Some properties of vorticity equation

By making the transformation $z = 1 - y$ in (2.14), it is seen that if $\psi(y; c)$ is a solution of the equation then so is $\psi(1 - y; 1 - c)$. Moreover, if $\psi(y; c)$ is a solution of (2.14) then its complex conjugate $\psi^*(y; c^*)$ is also a solution. Thus the solutions have Hermitian symmetry about $y = \frac{1}{2}$ such that if $\psi(y; c)$ is a solution then so is $\psi^*(1 - y; 1 - c^*)$. Drazin and Howard [3] show that this property, associated with antisymmetric basic profiles, sometimes implies that all unstable disturbances propagate at the mean flow speed. The analysis in § 5 and § 6 suggests that this situation prevails for (2.14).

When the eigenvalue c is complex, the solution of (2.14) is regular in the solution domain (0, 1) and so it may be written in the form

$$\psi'' - k^2\psi = \varepsilon^2\sigma^2\psi/\{(y - c)^2 - \varepsilon^2\}. \tag{4.1}$$

Multiplying (4.1) by ψ^*, (4.1)* by ψ and subtracting, we find that

$$(\psi^*\psi' - \psi\psi^{*\prime})' = i4c_i\varepsilon^2\sigma^2(y - c_r)|\psi|^2/|(y - c)^2 - \varepsilon^2|^2, \tag{4.2}$$

where $c = c_r + ic_i$ with c_r and c_i real. By integrating (4.2) and using the boundary conditions on ψ, we see that

$$c_i \int_0^1 dy(y - c_r)|\psi|^2/|(y - c)^2 - \varepsilon^2|^2 = 0.$$

Because $c_i \neq 0$, this implies that the phase speed of any unstable mode is equal to the mean velocity of the basic flow at some position in the channel; that is $c_r \in (0, 1)$. From this and the Hermitian symmetry of the solutions, we see that it is necessary to consider solutions for $c_r \in (0, \frac{1}{2}]$ only.

The primary effect of the fluctuations in the basic flow on the vorticity equation is the bifurcation of the singular point of the equation. The deterministic linear profile gives rise to a singular point at $y = c$, while the random basic profile has singular points at $y = c \pm \varepsilon$; that is, these points are displaced from $y = c$ by the root mean square velocity of the fluctuations. For a neutrally stable disturbance ($c_i = 0$), these singular points lie on the real y-axis and perhaps in the solution domain. Indeed, if

$$\varepsilon^2 < \min(c^2, (1 - c)^2) \tag{4.3}$$

then it is easy to show that at least one singular point must lie in (0, 1). If this is not the case then (4.1) is valid for all $y \in (0, 1)$. Multiplying (4.1) by ψ^*, integrating over (0, 1) and applying the boundary conditions gives

$$\int_0^1 dy\{|\psi'|^2 + [k^2 + \varepsilon^2\sigma^2/\{(y-c)^2 - \varepsilon^2\}]|\psi|^2\} = 0. \tag{4.4}$$

Clearly, the only solution of (4.4) satisfying (4.3) is $\psi \equiv 0$. Thus the assumption that (4.1) is valid over the whole solution domain must be invalid.

To investigate the behaviour of ψ near $y = c + \varepsilon$, we make the transformation

$$z = y - c - \varepsilon \tag{4.5}$$

and expand the solution about $z = 0$ using the method of Frobenius. It is found that $z = 0$ is a regular singular point and that ψ_1 and ψ_2 are two independent solutions of (2.14) where

$$\psi_1(z) = z + \tfrac{1}{4}\varepsilon\sigma^2 z^2 + \ldots,$$
$$\psi_2(z) = \psi_1(z)\ln z + 2/\varepsilon\sigma^2 + \tfrac{1}{4}(k^2/\sigma^2 - 1 - \tfrac{3}{2}\varepsilon^2\sigma^2)z^2/\varepsilon + \ldots. \tag{4.6}$$

The solution near $y = c - \varepsilon$ is given by replacing ε by $-\varepsilon$ in (4.5)–(4.6). In general, therefore, the solution of (2.14) has branch points at $y = c \pm \varepsilon$ and the precise form of ψ is determined by the manner in which the branch cuts from these points go to infinity. Now (2.6) and (2.7) imply that, because $\Psi_0 \equiv 0$ for $t < 0$, ψ is analytic in the upper half c-plane when k is positive. This condition is satisfied if the imaginary part of $y - c \pm \varepsilon$ increases as the modulus of $y - c \pm \varepsilon$ increases along the branch cuts, as shown in Figure 1.

Figure 1a depicts the situation for an unstable disturbance when the amplitude of the fluctuations in the basic flow is small; in particular, when $\varepsilon < \frac{1}{2}$. For a stable disturbance when $\varepsilon < \frac{1}{2}$, Figure 1b shows that the branch cuts intersect the interval (0, 1).In this case, the path of any integral over the solution domain $\{y \in (0, 1)\}$ must be distorted to pass under the branch points.

It is seen from Figure 1c that the radius of convergence of a Taylor series expansion of $\psi(y)$ about $y = \frac{1}{2}$ is larger than $\frac{1}{2}$ when the amplitude of the fluctuations in the basic flow is large ($\varepsilon > \frac{1}{2}$). Thus such a regular expansion can be used to describe any disturbance (stable or unstable) when $1 - \varepsilon < c_r < \varepsilon$. Similarly, a single expansion is valid over the whole solution domain when $c_i > \frac{1}{2}$, independently of ε: this is seen from Figures 1a and 1c. We now seek formal asymptotic solutions of (2.14) corresponding to these two situations.

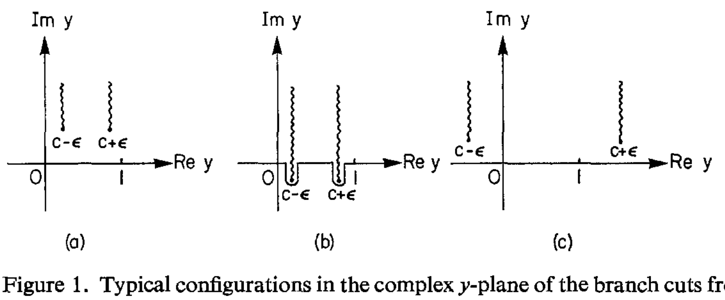

Figure 1. Typical configurations in the complex y-plane of the branch cuts from the singular points of (2.14). (a) $c_i > 0$, $\varepsilon < c_r < 1 - \varepsilon$; (b) $c_i \leqq 0$, $\varepsilon < c_r < 1 - \varepsilon$; (c) $1 - \varepsilon < c_r < \varepsilon$.

5. Asymptotic solution for large fluctuations

We consider the formal asymptotic solution of (2.14) as $\varepsilon \to \infty$ with σ fixed. It is seen from sec. 4 that such an expansion is uniformly valid for

$$\varepsilon^2 > \max(c_r^2, (1 - c_r)^2), \tag{5.1}$$

and so the expansion ought to be of practical use for $\varepsilon > \frac{1}{2}$. Thus the physical condition on (2.14) that ε^2 must be small compared with unity can be satisfied in practice. The expansion is also valid for all real values of c_i; stable and unstable modes occur as complex conjugates.

Equation (2.14) may be written as

$$\{1 + (c_i/\varepsilon)^2 + i2\varepsilon^{-1}(c_i/\varepsilon)(y - c_r)^2\}(\psi'' - k^2\psi) + \sigma^2\psi = 0, \text{ with } \psi = 0 \text{ on } y = 0, 1. \tag{5.2}$$

For the leading terms of (5.2) as $\varepsilon \to \infty$ to correspond to an eigenvalue problem, it is necessary to have c_i formally of order ε. We therefore take the following asymptotic power series expansions for ψ and c:

$$\psi \sim \sum_{n=0}^{\infty} \varepsilon^{-n}\psi_n(y), \quad c_r \sim \sum_{n=0}^{\infty} \varepsilon^{-n}c_r^{(n)}, \quad c_i \sim \sum_{n=0}^{\infty} \varepsilon^{1-n}c_i^{(n)}. \tag{5.3}$$

A sequence of equations for $\{\psi_n\colon n = 0, 1, \ldots\}$ is found by putting (5.3) into (5.2) and equating the coefficients of like powers of ε. The first three such equations are

$$\{1 + c_i^{(0)2}\}(\psi_0'' - k^2\psi_0) + \sigma^2\psi_0 = 0, \quad \psi_0 = 0 \text{ on } y = 0, 1; \tag{5.4}$$

$$\begin{aligned}\{1 + c_i^{(0)2}\}(\psi_1'' - k^2\psi_1) + \sigma^2\psi_1 &= -2c_i^{(0)}c_i^{(1)}(\psi_0'' - k^2\psi_0)\\ &- i2c_i^{(0)}\{y - c_r^{(0)}\}(\psi_0'' - k^2\psi_0), \quad \psi_1 = 0 \text{ on } y = 0, 1;\end{aligned} \tag{5.5}$$

$$\begin{aligned}\{1 + c_i^{(0)2}\}(\psi_2'' - k^2\psi_2) + \sigma^2\psi_2 &= -\{c_i^{(1)2} + 2c_i^{(2)}c_i^{(2)}\}(\psi_0'' - k^2\psi_0)\\ &- 2c_i^{(0)}c_1^{(1)}(\psi_1'' - k^2\psi_1) + \{y - c_r^{(0)}\}^2(\psi_0'' - k^2\psi_0)\\ &- i2c_i^{(0)}\{y - c_r^{(0)}\}(\psi_1'' - k^2\psi_1) + i2c_i^{(0)}c_r^{(1)}(\psi_0'' - k^2\psi_0)\\ &- i2c_i^{(1)}\{y - c_r^{(0)}\}(\psi_0'' - k^2\psi_0), \quad \psi_2 = 0 \text{ on } y = 0, 1.\end{aligned} \tag{5.6}$$

The solution of the zeroth order eigenvalue problem (5.4) is

$$\psi_0 = \sin n\pi y \ (n = 1, 2, \ldots) \text{ with } c_i^{(0)2} = \sigma^2/\{k^2 + (n\pi)^2\} - 1 > 0. \tag{5.7}$$

Now the Fredholm alternative implies that the right hand sides of (5.5) and (5.6) are orthogonal to ψ_0 over the interval (0, 1). Thus, putting (5.7) into (5.5) and using the condition that c_i and c_r are real, we find the eigenvalue relations

$$c_i^{(1)} = 0 \text{ and } c_r^{(0)} = \tfrac{1}{2}, \tag{5.8}$$

provided that $c_i^{(0)} \neq 0$. (The case of a neutral disturbance ($c_i = 0$) is treated separately below.) The first order correction to the eigenfunction is therefore found to be

$$\psi_1 = i\tfrac{1}{2}\sigma^2 c_i^{(0)}\{(y - \tfrac{1}{2})\sin n\pi y + [\tfrac{1}{4} - (y - \tfrac{1}{2})^2]n\pi \cos n\pi y\}/\{n\pi[1 + c_i^{(0)2}]\}^2. \tag{5.9}$$

By putting (5.7)–(5.9) into (5.6) and using the Fredholm alternative, it is seen that

$$c_r^{(1)} = 0 \text{ and } c_i^{(0)}c_i^{(2)} = \{\tfrac{1}{24} - 1/4(n\pi)^2\}\{-3 + 4[k^2 + (n\pi)^2]/\sigma^2\}$$
$$+ \{\tfrac{1}{24} - 5/8(n\pi)^2\}\{k^2 + (n\pi)^2\}\{1 - [k^2 + (n\pi)^2]/\sigma^2\}/(n\pi)^2.$$

It is apparent that $c_i^{(2n-1)} = 0$ and $c_r^{(n)} = 0$ for $n = 1, 2, \ldots$. Moreover, if ψ_{2n} is a real even (odd) function of $y - \frac{1}{2}$ then ψ_{2n-1} is an imaginary odd (even) function of $y - \frac{1}{2}$ and is proportional to $c_i^{(0)}$. Thus the asymptotic expansion of ψ assumes the Hermitian symmetry discussed in § 4. In particular, we find that as $\varepsilon \to \infty$ with σ fixed

$$\psi = \sin n\pi y + i[c_i^{(0)}/2\varepsilon\sigma^2(n\pi)^2]\{k^2 + (n\pi)^2\}^2\{(y - \tfrac{1}{2}) \sin n\pi y$$
$$+ n\pi[\tfrac{1}{4} - (y - \tfrac{1}{2})^2] \cos n\pi y\} + O(\varepsilon^{-2}), \tag{5.10}$$

where $c_r = \frac{1}{2}$ and

$$c_i^2 = \varepsilon^2\sigma^2/\{k^2 + (n\pi)^2\} - \varepsilon^2 - \tfrac{1}{4} + 3/2(n\pi)^2$$
$$+ \{\tfrac{1}{3} - 2/(n\pi)^2\}\{k^2 + (n\pi)^2\}/\sigma^2 + \{(n\pi)^2 - 15\}\{k^2 + (n\pi)^2\}$$
$$\times \{1 - [k^2 + (n\pi)^2]/\sigma^2\}/12(n\pi)^4 + O(\varepsilon^{-2}).$$

All unstable solutions of (2.14) satisfying the condition (5.1) are represented by (5.10), and so all such disturbances propagate at the average flow speed across the channel (i.e. $c_r = \frac{1}{2}$). Moreover, because $c_i^{(0)2}$ is positive, it is seen from (5.7) that solutions exist only if $\sigma > \pi$; that is, only if the correlation scale of the fluctuations in the basic flow is less than about 0.3 of the channel width. The number of modes that exist for a given correlation scale is equal to the largest integer smaller than σ/π. This implies that an unstable disturbance cannot be supported by the fluctuations in the basic flow once its lateral scale $(1/n)$ becomes smaller than the effective correlation scale (π/σ).

Thus, when $\varepsilon > \frac{1}{2}$ and $\sigma > \pi$, the asymptotic form for large t of an arbitrary initial disturbance consists of a finite number of non-dispersive modes propagating at the average flow speed. The growth rate of the spectral component with wavenumber k within the nth mode is

$$\Omega_n(k) = kc_i. \tag{5.11}$$

We see from (5.10) and (5.11) that each mode has a high wavenumber cut-off k_c where

$$k_c = \{\sigma^2 - (n\pi)^2\}^{\frac{1}{2}} + O(\varepsilon^{-2}).$$

The maximum growth rate for the nth mode is

$$\Omega_n(k_m) = \varepsilon(\sigma - n\pi) + O(1) \tag{5.12}$$

and this occurs at the wavenumber

$$k_m = \{n\pi(\sigma - n\pi)\}^{\frac{1}{2}} + O(\varepsilon^{-2}).$$

Hence the first mode ($n = 1$) dominates the asymptotic state of any initial disturbance, and (5.12) shows that the asymptotic growth rate increases with increasing $\varepsilon\sigma$, the root mean square strain rate of the fluctuations in the basic flow. Figure 2 shows the behaviour of $\Omega_1(k)$ given by (5.10) and (5.11) as the root mean square velocity of the fluctuations (ε) increases from $\frac{1}{2}$ to 1 when $\sigma = 20$.

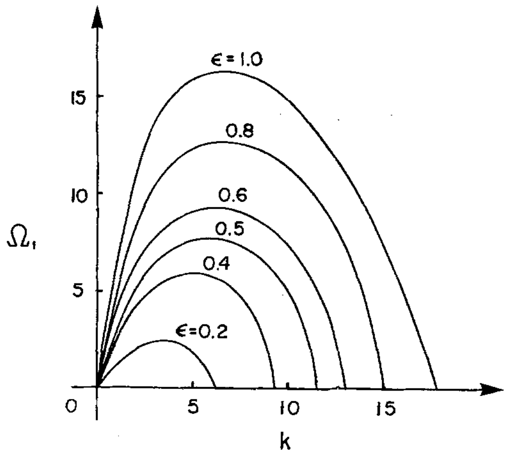

Figure 2. Behaviour of growth rate $\Omega_1(k)$ of the first mode disturbance when $\sigma = 20$. Numbers refer to ε.

The solution (5.10) is found under the assumption that $c_i^{(0)}$ is non-zero. To investigate the asymptotic behaviour of neutral disturbances as $\varepsilon \to \infty$ with σ fixed, we set $c_i = 0$ in (5.2). Now the eigenfunctions are real, in addition to being regular, provided that (5.1) is satisfied. The appropriate form for the eigenvalue relation is given by k^2 as a function of the real phase speed c, and so the streamfunction ψ and wavenumber k are expanded in the form

$$\psi \sim \sum_{n=0}^{\infty} \varepsilon^{-2n}\psi_n(y), \; k^2 \sim \sum_{n=0}^{\infty} \varepsilon^{-2n}k_n^2. \tag{5.13}$$

By putting (5.13) into (5.2) with $c_i = 0$, it is readily found that

$$\begin{aligned}\psi = {} & \{1 - \sigma^2(y-c)^2/4\varepsilon^2(n\pi)^2\} \sin n\pi y + \{\sigma^2/6\varepsilon^2 n\pi\}\{(y-c)^3 \\ & - 3[(\tfrac{1}{2}-c)^2 + \tfrac{1}{12}](y-c) - c(1-c)(1-2c)\} \cos n\pi y + O(\varepsilon^{-4}), \\ & n = 1, 2, \ldots,\end{aligned} \tag{5.14}$$

and

$$k^2 = \sigma^2 - (n\pi)^2 + (\sigma^2/\varepsilon^2)\{(\tfrac{1}{2}-c)^2 + \tfrac{1}{12} - 1/2(n\pi)^2\} + O(\varepsilon^{-4}).$$

Thus, although large fluctuations in the basic velocity profile support only unstable disturbances which propagate at the average flow speed, neutral disturbances with any phase speed satisfying (5.1) can be found. We note that the eigenvalue relation in (5.14) with $c = \frac{1}{2}$ is consistent with that obtained by setting $c_i^{(0)} = 0$ and $c_i = 0$ in the eigenvalue relation of (5.10).

6. Asymptotic solution for small correlation scale

An asymptotic representation of the unstable solutions of (2.14) when $\varepsilon > \frac{1}{2}$ is given by the regular expansion (5.10). When $\varepsilon < \frac{1}{2}$, the real part of at least one of the singular points of (2.14) lies in the interval (0, 1) (see Figure 1) and so a regular expansion of the solution does not exist in general. On the other hand, if $c_i > \frac{1}{2}$ then a Taylor series expansion about

$y = \frac{1}{2}$ is valid over (0, 1). Hence we now look for an asymptotic solution of (2.14) corresponding to very unstable disturbances. This is done formally by writing (2.14) in the form

$$\{c_i^2 + i2c_i(y - c_r) + \varepsilon^2 - (y - c_r)^2\}(\psi'' - k^2\psi) + \varepsilon^2\sigma^2\psi = 0,$$
$$\text{with } \psi = 0 \text{ on } y = 0, 1, \tag{6.1}$$

and finding an expansion as $\sigma \to \infty$ with ε^2 fixed; that is, we seek a solution when the correlation scale of the fluctuations in the basic flow is very small. For the leading terms in (6.1) to represent an eigenvalue problem, c_i must be of order σ. We therefore take

$$\psi \sim \sum_{n=0}^{\infty} \sigma^{-n}\psi_n(y), \; c_r \sim \sum_{n=0}^{\infty} \sigma^{-n}c_r^{(n)}, \; c_i \sim \sum_{n=0}^{\infty} \sigma^{1-n}c_i^{(n)}. \tag{6.2}$$

By putting (6.2) into (6.1), a sequence of equations for the $\{\psi_n: n = 0, 1, \ldots\}$ is found. These equations may be solved as in § 5 and we then obtain

$$\psi = \sin n\pi y + i\{[k^2 + (n\pi)^2]/2\sigma c_i^{(0)}(n\pi)^2\}\{(y - \tfrac{1}{2}) \sin n\pi y$$
$$+ n\pi[\tfrac{1}{4} - (y - \tfrac{1}{2})^2] \cos n\pi y\} + 0(\sigma^{-2}), \; n = 1, 2, \ldots, \tag{6.3}$$

where $c_r = \frac{1}{2}$,

$$c_i^2 = \varepsilon^2\sigma^2/\{k^2 + (n\pi)^2\} - \varepsilon^2 - \tfrac{1}{4} + 3/2(n\pi)^2$$
$$- \{15 - (n\pi)^2\}\{k^2 + (n\pi)^2\}/12(n\pi)^4 + 0(\sigma^{-4}),$$
$$c_i^{(0)2} = \varepsilon^2/\{k^2 + (n\pi)^2\}.$$

The expansion (6.3) ought to be valid for all values of ε provided that $c_i > \frac{1}{2}$. Indeed, we see that the solution (5.10) for large ε reduces to (6.3) in the limit $\sigma \to \infty$. These two solutions suggest strongly that all unstable disturbances propagate at the average flow speed in the channel; i.e. $c_r = \frac{1}{2}$ for all $c_i > 0$. The only unstable disturbances not described by either (5.10) or (6.3) occur when both ε and σ are small. In particular, the condition that the leading term for c_i in (6.3) is greater than $\frac{1}{2}$ implies that the root mean square strain rate of the fluctuations ($\varepsilon\sigma$) must be greater than $\pi/2$.

It is seen from (6.3) that, for a fixed r.m.s. strain rate $\varepsilon\sigma$, the growth rate of a disturbance decreases with increasing r.m.s. velocity ε. Thus the maximum growth rate occurs as $\varepsilon \to 0$ (and so $\sigma \to \infty$) which corresponds mathematically to the coalescence of the two singular points of (2.14).

As for the expansion in § 5 when $\varepsilon > \frac{1}{2}$, only a finite number of modes exist for a given σ and ε. However, the first mode ($n = 1$) is the fastest growing disturbance and so it dominates the asymptotic form of the solution of (2.4) for large t. The behaviour of the growth rate $\Omega_1(k)$ for $\varepsilon < \frac{1}{2}$ and $\sigma = 20$, calculated from (5.11) and (6.3), is shown in figure 2.

Acknowledgements

This work was carried out while one of us (M.J.M.) was visiting the Department of Mathematics, University of British Columbia. The support of the National Research Council of Canada, through Operating Grant A5201, is gratefully acknowledged.

REFERENCES

[1] K. M. Case, Stability of inviscid plane Couette flow, *Physics of Fluids*, 3 (1960) 143–148.
[2] R. Betchov and W. O. Criminale, *Stability of Parallel Flows*, Academic Press (1967).
[3] P. G. Drazin and L. N. Howard, Hydrodynamic stability of parallel flow of inviscid fluid, in *Advances in Applied Mechanics*, 9 (1966) 1–89, Academic Press.
[4] U. Frisch, Wave propagation in random media, in *Probabilistic Methods in Applied Mathematics*, 1 (1968) 75–198, A. T. Bharucha-Reid (editor), Academic Press.
[5] M. S. Howe, Wave propagation in random media, *Journal of Fluid Mechanics*, 45 (1971) 769–783.